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Strongly coupled CFT in FRW universe from AdS/CFT correspondence

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Abstract: We develop a formalism to calculate the effective action of the strongly coupled conformal field theory (CFT) in curved spacetime. The effective action of the CFT is obtained from AdS/CFT correspondence. The anti de-Sitter (AdS) spacetime has various slicing which give various curved spacetime on its boundary. We show the de Sitter spacetime and the Friedmann-Robertson-Walker (FRW) universe can be embedded in the AdS spacetime and derive the scalar two-point function of the conformal fields in those spacetime. In curved spacetime, the two-point function depends on the vacuum state of the CFT. A method to specify the vacuum state in AdS/CFT calculations is shown. Because the classical action in AdS spacetime diverges near the boundary, we need the counter terms to regulate the result. The simple derivation of the counter terms using the Hamilton-Jacobi equation is also presented in the appendix.

Keywords: Cosmology of Theories beyond the SM, Physics of the Early Universe.

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1. Introduction

There are many matter fields in the Standard Model of particle physics. Supersymmetric theory or string theory predicts much larger number of matter fields. In the early universe, these matter fields are effectively massless and then behave as the conformaly invariant fields. Quantum effects of them can play an important role in the history of the universe. One of the examples is the trace anomaly of the energy-momentum tensor of the matter fields. The trace anomaly acts like a cosmological constant and could lead to the inflationary universe [1, 2].

In order to know the quantum effects of the matter fields, we should know the effective action which describes the effects of the matter fields. A way of calculating the effective action is to use AdS/CFT correspondence [2, 3]. AdS/CFT correspondence states quantum theory of CFT on d-dimensional boundary of the Anti-de Sitter (AdS) spacetime can be described by the classical theory in the $(d+1)$-dimensional AdS bulk. This correspondence is on the ground of duality in the string theory. The classical limit of type IIB superstring theory (supergravity) on five dimensional AdS spacetime times a five sphere ($AdS_5 \times S^5$) is dual to $\mathcal{N} = 4$ supersymmetric

SU(N) Yangs- Mills (YM) theory for large N. The CFT parameters are related to the supergravity parameters by $l^3/G = 2N^2/\pi$, where G is the 5-dimensional Newton constant and l is the curvature radius of AdS_5. The curvature scale of the AdS_5 spacetime l is given by $l^2 = \sqrt{g_{YM}^2 N}\, l_s^2$ where g_{YM} is the coupling constant of YM theory and l_s is the string length. In order to trust the classical theory in the bulk we need $(l/l_s)^4 = g_{YM}^2 N \gg 1$. Thus, the dual CFT should have large N.

Practically, the AdS/CFT correspondence gives an efficient way of calculating a non-local effective action in the boundary spacetime. Let us consider the gravitational theory. In the dual theory on the boundary, gravity is coupled to the matter fields. By performing the path integral over the matter fields, an effective action that is the functional of the background metric is obtained. In general, the effective action contains non-local terms which make the calculations difficult. If one uses the AdS/CFT correspondence, the non-local effective action can be found from the classical action in the AdS_{d+1} spacetime. Indeed, quantum corrections of the matter fields on the graviton propagator were calculated using an effective action derived from AdS/CFT correspondence and the result was shown to be agreed with the old calculations done in d-dimensional quantum field theory [4]. So far much of the calculations have been done in the flat spacetime. In order to observe the effects of the quantum matter in the early universe, we should know the effective action in curved spacetime. Several authors extended the calculations to the de Sitter spacetime [2], [5]–[7]. We will extend their works to several directions.

The main claim of our paper is that the effective action in curved spacetime can be constructed by choosing appropriate slicing in the AdS bulk. The AdS spacetime has various slicing which give various curved spacetime on its boundary. For example, the de Sitter spacetime and the Friedmann-Robertson-Walker (FRW) universe can be embedded into the AdS spacetime. We will explicitly derive the two-point function of the conformal field in the de Sitter spacetime and the FRW spacetime. In curved spacetime, the two-point function of the quantum field depends on the choice of the vacuum. A method to specify the vacuum state of the CFT in AdS/CFT calculations will be presented.

We concentrate our attention on the scalar field theory in the bulk. In the dual theory on the boundary, there is a scalar field ϕ coupled with conformal matter fields ψ, say

$$S[\phi, \psi] = \int d^d x \sqrt{-h} \left(-\partial_\mu \phi \partial^\mu \phi + e^{-2\phi} \mathcal{L}_{\rm matter}[\psi] \right), \tag{1.1}$$

where $\mathcal{L}_{\rm matter}$ is the lagrangian for conformal matter fields ψ and h is the metric of the d-dimensional spacetime. As in the gravitational theory, the generating functional can be obtained by performing the path integral over the matter fields ψ to be a functional of the scalar field

$$W[\phi] = \int \mathcal{D}\psi e^{-iS[\phi,\psi]}, \tag{1.2}$$

AdS/CFT correspondence gives it as the generating functional of the dual scalar operator $\mathcal{O}$ coupled with the scalar field ϕ on the boundary

$$W[\phi] = W_{CFT} = \left\langle \exp\left(i \int d^d x \sqrt{-h} \phi \mathcal{O} \right) \right\rangle = \tag{1.3}$$
$$= \exp\left(i \int d^d x \sqrt{-h} \phi \langle \mathcal{O} \rangle - \frac{1}{2} \int d^d x d^d x' \sqrt{-h} \sqrt{-h} \phi(x) \phi(x') \langle \mathcal{O}(x) \mathcal{O}(x') \rangle + \cdots \right).$$

Varying the generating functional n times with respect to ϕ gives the n-point function of operator $\mathcal{O}$.

The outline of our paper is as follows. In section 2, we review the method to calculate the effective action in flat spacetime using the invariant Green function in AdS spacetime. In section 3, we extend the calculation to the de Sitter spacetime. We will explain how the information of the vacuum state of the boundary theory is encoded in the AdS/CFT calculation. Then the nature of the CFT in de Sitter spacetime is presented. Section 4 is devoted to the formulation for the FRW universe. In the appendix, the derivation of the counter terms via hamiltonian-Jacobi equation is shown, which are needed to regulate the action on the boundary.

2. A review

AdS/CFT correspondence states the generating function of the CFT W_{CFT} in d-dimensional spacetime can be obtained by the classical theory in the d+1-dimensional AdS spacetime AdS_{d+1} [3]. The correspondence is written as

$$W_{CFT} = \left\langle \exp\left(i \int_{\mathcal{B}} \phi_B \mathcal{O} \right) \right\rangle = e^{iI[\phi_B]}, \tag{2.1}$$

where $\mathcal{O}$ is the operator of the CFT coupled with the scalar field ϕ_B in the d-dimensional boundary $\mathcal{B}$ of the AdS_{d+1}. $I[\phi_B]$ is obtained from $d+1$-dimensional action of the scalar field ϕ in AdS_{d+1}

$$S[\phi] = -\frac{1}{2} \int d^{d+1} X \sqrt{-g} \left(\partial_A \phi \partial^A \phi + m^2 \phi^2 \right), \tag{2.2}$$

by inserting the solution of the classical field equation which approaches ϕ_B on the boundary. For simplicity, we concentrate our attention on the generating function of the two-point function. The extension to n-point function can be easily carried out [8]. The calculation of $I[\phi_B]$ is done as follows. First we should find the solution of the field equation in AdS_{d+1} spacetime which approaches ϕ_B on the boundary $\mathcal{B}$. The solution can be constructed from the Dirichlet Green function, which satisfies

$$(\Box_{d+1} - m^2) G(X; X') = -\frac{\delta^{d+1}(X - X')}{\sqrt{-g}},$$
$$G(X; X')|_{\mathcal{B}} = 0, \tag{2.3}$$

where $\Box_{d+1}$ is the laplacian operator in the AdS_{d+1} spacetime and X is the coordinate of AdS_{d+1}.

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From the general formalism of the Green function, the solution satisfying the field equation and boundary condition $\phi|_{\mathcal{B}} = \phi_B$ can be written as

$$\phi(X) = -\int_{\mathcal{B}} d^d x' \sqrt{-\gamma} G_{\mathcal{B}}(X;x')\phi_B(x') , \qquad G_{\mathcal{B}} = \left(n^A \frac{\partial}{\partial X'^A} G(X;X') \right)_{\mathcal{B}} , \quad (2.4)$$

where γ is the induced metric on the boundary $\mathcal{B}$, n^A is the unit vector normal to $\mathcal{B}$ and x is the coordinate of the boundary. $G_{\mathcal{B}}(X;x')$ is often called bulk-boundary propagator. Our task is to find the Dirichlet Green function in AdS_{d+1}. One way is to use the invariant Green function that depends only on the geodesic distance of the AdS spacetime [9]. The invariant Green function is defined as

$$G(X,X') = G(\mu(X,X')) , \quad (2.5)$$

where $\mu(X,X')$ is the geodesic distance between X and X'. For $X \neq X'$, $G(\mu)$ should satisfy the field equation $(\Box_{d+1} - m^2)G(\mu) = 0$ which can be rewritten in terms of μ as

$$\frac{d^2G(\mu)}{d\mu^2} + (d/l)\coth(\mu/l)\frac{dG(\mu)}{d\mu} - m^2 G(\mu) = 0 , \quad (2.6)$$

where l is the curvature radius of the AdS spacetime. After defining the new variable

$$W = \frac{2}{\cosh(\mu/l) + 1} , \quad (2.7)$$

the equation (2.6) can be converted into a hypergeometric equation. The equation (2.6) has two independent solutions. Among them, the solution which satisfies the Dirichlet boundary condition at the boundary $G(\mu)|_{\mathcal{B}} = 0$ is given by

$$G(\mu) = G_0 l^{1-d} W^{\Delta} F\left(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}, 2\Delta - d + 1, W \right) , \quad (2.8)$$

where F is the hypergeometric function,

$$\begin{aligned} G_0 &= \frac{i\Gamma[\Delta]}{\pi^{d/2} 2^{2\Delta+1} \Gamma[\Delta - d/2 + 1]} , \\ \Delta &= \frac{1}{2}(d + \sqrt{d^2 + 4m^2 l^2}) . \end{aligned} \quad (2.9)$$

The normalization G_0 is determined so as to have the same singularity with the Green function in flat spacetime for $\mu \to 0$

$$\lim_{\mu \to 0} G(\mu) = \frac{i\Gamma\left[(d+1)/2\right]}{2(d-1)\pi^{(d+1)/2}} \mu^{1-d} . \quad (2.10)$$

At the boundary of the AdS spacetime, $\mu \to \infty$ and $W \to 0$. The invariant Green function near the boundary is given by

$$G(\mu) \to G_0 W^{\Delta} , \quad (2.11)$$

which vanishes at the boundary as expected. Integrating by parts and using the equation of motion, the action $S[\phi]$ is rewritten as

$$S[\phi] = -\frac{1}{2}\int_{\mathcal{B}} d^d x \sqrt{-\gamma} n^A \phi \partial_A \phi \,, \tag{2.12}$$

where we have used the fact the Green function (2.8) vanishes at the horizon and then surface term at the horizon does not contribute to the action. Inserting the solution (2.4) into the action (2.12), we find

$$\begin{aligned} I[\phi_B] = \frac{1}{2}\int_{\mathcal{B}} & d^d x d^d x' \phi_B(x)\phi_B(x') \times \\ & \times \left(\sqrt{-\gamma(x)}\sqrt{-\gamma(x')} n^A(X) n^B(x') \frac{\partial^2}{\partial X^A \partial X'^B} G(X;X')\right)_{\mathcal{B}} . \end{aligned} \tag{2.13}$$

So far, we do not specify the geometry of the boundary. In AdS spacetime various spacetime can be embedded. The most simple one is the Minkowski spacetime. We use the Poincare coordinate to represent the AdS_{d+1}

$$ds^2 = \left(\frac{l}{z}\right)^2 (dz^2 - d\tau^2 + \delta_{ij} dx^i dx^j) \,. \tag{2.14}$$

Each slicing of $z = const.$ and the boundary $\mathcal{B}$ are Minkowski spacetime. Because the boundary is located at $z = 0$, we first consider the slicing $z = \epsilon$ and then take the limit $\epsilon \to 0$. The invariant W can be expressed in terms of the Poincare coordinate as

$$W = 4\frac{zz'}{(z+z')^2 + |x-x'|^2} \,, \tag{2.15}$$

where $|x-x'|^2 = -(\tau-\tau')^2 + |x^i - x'^i|^2$. Then from (2.11), we can show

$$\lim_{z=z'=\epsilon\to 0} \frac{\partial^2}{\partial z \partial z'} G(z,x;z'x') = G_0 l^{1-d} 2^{2\Delta} \Delta^2 \epsilon^{2(\Delta-1)} \frac{1}{|x-x'|^{2\Delta}} \,. \tag{2.16}$$

Because $\sqrt{-\gamma} = (\epsilon/l)^{-d}$ and $n^A(\partial/\partial X^A) = -(\epsilon/l)\partial/\partial z$, we find

$$I[\phi_B] = \frac{1}{2}\int_{\mathcal{B}} d^d x d^d x' \phi_B(x)\phi_B(x') \epsilon^{2(\Delta-d)} l^{d-1} \left(G_0 2^{2\Delta}\Delta^2 \frac{1}{|x-x'|^{2\Delta}}\right). \tag{2.17}$$

Then renormalizing ϕ_B as

$$\tilde{\phi}_B = \frac{\Delta}{2\Delta - d} \epsilon^{\Delta-d} l^{(d-1)/2} \phi_B \,, \tag{2.18}$$

the two point function of the CFT is obtained from (1.3) and (2.1) as

$$\begin{aligned} \langle \mathcal{O}(x)\mathcal{O}(x')\rangle &= \frac{(-i)\delta^2 I}{\delta\tilde{\phi}_B(x)\delta\tilde{\phi}_B(x')} = C_2 \frac{1}{|x-x'|^{2\Delta}} \,, \\ C_2 &= 2\left(\Delta - \frac{d}{2}\right) \frac{\Gamma[\Delta]}{\pi^{d/2}\Gamma[\Delta - d/2]} \,. \end{aligned} \tag{2.19}$$

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The need of the factor $\Delta/(2\Delta - d)$ comes from the limiting procedure. We first take the limit $\epsilon \to 0$ in constructing the bulk-boundary propagator (2.4). That is, we used the Dirichlet Green function which vanishes at $z = 0$ and not at $z = \epsilon$. More detailed discussions are seen in [10]

3. CFT in de Sitter spacetime

3.1 CFT in de Sitter spacetime

The AdS spacetime has various slicing which realize various curved spacetime on it. These slicing can be obtained by the coordinate transformation from the Poincare coordinate. Let us consider the coordinate transformation

$$z = \eta \sinh y \,, \qquad \tau = \eta \cosh y \,. \tag{3.1}$$

The metric (2.14) becomes

$$ds^2 = \left(\frac{l^2}{\sinh^2 y}\right)\left(dy^2 + \frac{1}{\eta^2}(-d\eta^2 + \delta_{ij}dx^i dx^j)\right) . \tag{3.2}$$

On the slicing $y = y_0$, the embedded spacetime is the de-Sitter spacetime with the Hubble parameter $H = l^{-1}\sinh y_0$.

If we use the invariant Green function in calculating the bulk-boundary propagator, the calculation of the generating function is essentially the same with the Minkowski spacetime. We should merely write the invariant length in terms of the associated coordinate. One subtle point is the procedure of taking the limit to the boundary because the de Sitter spacetime with specific H is realized on only one slicing $y = y_0$ in the AdS spacetime. We need to introduce the parameter ϵ which controls the limit to the boundary, $z \propto \epsilon$. We can define

$$\epsilon = \sinh y_0 \,. \tag{3.3}$$

Note that because $\sinh y_0 = Hl$, we should take $l \to 0$ to fix H in taking the limit $\epsilon \to 0$. Once the limiting procedure is established, the effective action can be obtained using (2.13). W can be expressed in terms of the coordinate (3.2) as

$$W = 4\frac{\eta\eta' \sinh y \sinh y'}{-(\eta^2 + \eta'^2 - 2\eta\eta' \cosh(y + y')) + |x^i - x'^i|^2} . \tag{3.4}$$

Then we find

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\partial^2}{\partial y \partial y'}G(y,x;y'x') &= G_0 2^{2\Delta}\Delta^2\epsilon^{2(\Delta-1)}\left(\frac{\eta\eta'}{-(\eta-\eta')^2 + |x^i - x'^i|^2}\right)^{\Delta} \\ &= G_0 2^{2\Delta}\Delta^2\epsilon^{2(\Delta-1)}\sigma^{-2\Delta} \,, \end{aligned}\tag{3.5}$$

where σ is the invariant length of the de Sitter spacetime. Denoting the metric of the de Sitter spacetime as

$$h_{\mu\nu}dx^\mu dx^\nu = \frac{1}{\eta^2}(-d\eta^2 + \delta_{ij}dx^i dx^j)\,, \tag{3.6}$$

we have $\sqrt{-\gamma} = (\epsilon/l)^{-d}\sqrt{-h}$. Then (2.13) becomes

$$I[\phi_B] = \frac{1}{2}\int_{\mathcal{B}} d^dx d^dx' \sqrt{-h(x)}\sqrt{-h(x')}\phi_B(x)\phi_B(x')\epsilon^{2(\Delta-d)}l^{d-1}G_0 2^{2\Delta}\Delta^2\sigma^{-2\Delta}\,. \tag{3.7}$$

Renormalizong ϕ_B as in (2.18), the two point function of the CFT in de Sitter spacetime is obtained as

$$\langle\mathcal{O}(x)\mathcal{O}(x')\rangle_{dS} = C_2\sigma^{-2\Delta}. \tag{3.8}$$

3.2 Vacuum state of CFT

In the previous subsection, we obtained the generating functional of the CFT from invariant Green function of AdS_{d+1} spacetime. In curved spacetime, the correlation function of the CFT depends on the choice of the vacuum. In the AdS/CFT correspondence, the vacuum state on the boundary theory corresponds to the boundary condition of the bulk-boundary propagator [11]–[13]. Using the invariant Green function in constructing the bulk-boundary propagator will specify the vacuum state of the CFT.

To see this fact, it is convenient to do the calculations in momentum spacetime. The bulk-boundary propagator in the momentum space can be constructed from the solution of the field equation in AdS_{d+1}.

$$(\Box_{d+1} - m^2)\phi(y,x) = 0\,. \tag{3.9}$$

In the coordinate (3.2), the solution can be separated with respect to the coordinate of the bulk y and the d-dimensional boundary x,

$$\phi(y,x) = \int dp'\, f_{p'}(y)Y_{p'}(x)\,. \tag{3.10}$$

Here we introduced the scalar harmonics $Y_{p'}$ in the d-dimensional de Sitter spacetime. We notice that there are two ambiguities of the boundary conditions in the solution. One is the boundary condition at the horizon of the AdS_{d+1} spacetime which specifies $f_{p'}(y)$. Another is the choice of the time slicing in the d-dimensional spacetime which is determined by the choice of the harmonics $Y_{p'}(x)$. The freedom of the choice in $f_{p'}(y)$ may correspond to the freedom to choose different lorentzian propagators in the CFT (Feynman propagator, retarded propagator etc.) [11]. The choice of the time slicing will lead to the choice of the vacuum state in the CFT in curved spacetime. So, choosing an appropriate harmonics, we can find the CFT in a desired vacuum.

Now the question is which boundary condition leads to the result (3.8). We will show taking the euclidean boundary condition gives the result. For computational simplicity, we assume a closed de Sitter spacetime and take $d = 4$. Then, we can perform the calculations in the euclidean spacetime in which the spacetime is four sphere S^4 [2, 5]. We first solve the field equation in the bulk. The field equation is given by

$$\left(\partial_y^2 - 3\coth y\partial_y + \left(p'^2 + \frac{9}{4}\right) - \frac{m^2l^2}{\sinh^2 y}\right) f_{p'}(y) = 0\,,$$
$$\Box_4 Y_{p'}(x) = \left(p'^2 + \frac{9}{4}\right) Y_{p'}(x)\,. \quad (3.11)$$

Making the analytic continuation to euclidean spacetime and putting

$$p' = i\left(p + \frac{3}{2}\right), \quad (3.12)$$

we obtain the solutions of the field equation (3.11)

$$\begin{aligned} f_p(y) &= (\sinh y)^2 Q^{\Delta-2}_{p+1}(\cosh y)\,, \\ \Box_4 Y_p(x) &= -p(p+3)Y_p(x)\,, \end{aligned} \quad (3.13)$$

where Q is the associated Legendre function of the second kind, $\Delta = 2+\sqrt{4+m^2l^2}$, $Y_p(x)$ is the scalar harmonics on S^4. From euclidean boundary condition, we have chosen the solution $f_p(y)$ which is regular at $y \to \infty$. Using these mode functions, we can write the solution of the field equation with boundary condition $\phi(y = y_0) = \phi_B$ as

$$\phi(y, x) = \sum_p \frac{f_p(y)}{f_p(y_0)} Y_p(x)\phi_B(p)\,. \quad (3.14)$$

Then the generating function can be obtained by inserting the solution (3.14) into (2.12) and using the orthonormal relation of the harmonics,

$$\int d^4x\sqrt{h}Y_p(x)Y^*_{p'}(x) = \delta_{pp'}\,. \quad (3.15)$$

We obtain

$$\begin{aligned} I[\phi_B] &= \sum_p \lim_{\epsilon\to 0}[\epsilon^{-3}K_p(\epsilon)]\phi_B(p)\phi_B(-p)\,, \\ K_p[\epsilon] &= -\frac{1}{2}\frac{f'(y_0)}{f(y_0)}\,, \end{aligned} \quad (3.16)$$

where the field is rescaled as $\phi_B \to l^{-3/2}\phi_B$. There is a problem in taking the limit $\epsilon \to 0$ because $K_p(\epsilon)$ diverges as $\epsilon \to 0$. We should introduce the counter terms to regulate the result. Since the structure of the divergence is different for massless and massive field, we will treat them separately.

First we consider the massless case. $K_p(\epsilon)$ can be expanded in terms of ϵ as

$$\begin{aligned}\epsilon^{-3}K_p(\epsilon) = &\frac{1}{4}p(p+3)\epsilon^{-2} + \frac{1}{8}p(p+1)(p+2)(p+3)\log\epsilon + \\ &+\frac{1}{8}\left(-p(p+3) + p(p+1)(p+2)(p+3)(\psi(p+2)+\gamma-\log 2)\right) + \cdots ,\end{aligned} \tag{3.17}$$

where ψ is the poli-gamma function and γ is the Euler number. The first two terms diverge as $\epsilon \to 0$ and these terms should be canceled by the counter terms. The covariant forms of the counter terms are calculated in the Appendix using the Hamilton-Jacobi equation. The counter terms are given by

$$e^{S_{CT}} = e^{S^{(2)}+S^{(4)}}, \tag{3.18}$$

$$S^{(2)} = -\frac{1}{4}\epsilon^{-2}\int d^4x\sqrt{h}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,, \tag{3.19}$$

$$S^{(4)} = -\frac{1}{8}\log\epsilon\int d^4x\sqrt{h}\left(\frac{2}{3}R\partial_\mu\phi\partial^\mu\phi - 2R_{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (\Box_4\phi)^2\right), \tag{3.20}$$

where we have made analytic continuation of the result obtained in the appendix (A.40) to euclidean spacetime. At finite order of ϵ, terms which are analytic in p can be canceled by a local counter term. We will consider the term which cannot be canceled by a local counter term. The correpondance in euclidean spcaetime is given by

$$e^{-I[\phi_B]} = \left\langle \exp\left(\int d^4x\sqrt{h}\mathcal{O}\phi_B\right)\right\rangle . \tag{3.21}$$

Then, the two point function of CFT can be obtained by varying I twice with respect to ϕ_B as

$$\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle = -\frac{1}{4}p(p+1)(p+2)(p+3)\psi(p+2)\,. \tag{3.22}$$

Next consider the massive field. K_p can be expanded as

$$\begin{aligned}\epsilon^{-3}K_p = &\frac{1}{2}\epsilon^{-4}(\Delta-4) + \frac{1}{4}\epsilon^{-2}\left(\frac{p(p+3)}{\Delta-3} - \frac{2(\Delta-4)}{\Delta-3}\right) + \\ &+\frac{1}{2}\epsilon^{2(4-\Delta)}2^{-2\Delta+5}\frac{\Gamma(3-\Delta)\Gamma(p+\Delta)}{\Gamma(\Delta-2)\Gamma(p-\Delta+4)}\cdots .\end{aligned} \tag{3.23}$$

Again, the first two terms diverge as $\epsilon \to 0$. These divergences can be canceled by the counter terms

$$\begin{aligned}S^{(0)} &= \frac{1}{2}\epsilon^{-4}\int d^4x\sqrt{h}(4-\Delta)\phi^2\,, \\ S^{(2)} &= -\frac{1}{4(\Delta-3)}\epsilon^{-2}\int d^4x\sqrt{h}\left(\frac{4-\Delta}{6}R\phi^2 + h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right),\end{aligned} \tag{3.24}$$

where we have made analytic continuation of the result in the appendix (A.41) to euclidean spacetime. There is a finite term of the order ϵ^0. Because it is analytic in

p ($\propto (p+4-\Delta)(p-1+\Delta)(p+1)(p+2)$), it can be canceled by a local counter term. The leading term which is non-analytic in p is the last term in (3.23). Rescaling the source function $\phi_B \to \epsilon^{4-\Delta}\phi_B$, the two point function of the CFT becomes

$$\langle \mathcal{O}(p)\mathcal{O}(-p)\rangle = -2^{-2\Delta+5}\frac{\Gamma(3-\Delta)\Gamma(p+\Delta)}{\Gamma(\Delta-2)\Gamma(p-\Delta+4)}\,. \tag{3.25}$$

The correlation function in the real spacetime is obtained by mode summation of harmonics of S^4 [14]

$$\sum_{lmn} Y_{\rm plmn}(x)Y^*_{\rm plmn}(x') = \frac{1}{4\pi^{5/2}}(2p+3)\Gamma\left(\frac{3}{2}\right)C_p^{3/2}(\cos\gamma_4)\,, \tag{3.26}$$

and the formula [15]

$$\sum_p \frac{\Gamma(p+\Delta)}{\Gamma(p-\Delta+4)}\left(p+\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)C_p^{3/2}(\cos\gamma_4) = \frac{\Gamma(\Delta)}{\Gamma(2-\Delta)}2^{\Delta-3}\sqrt{\pi}(1-\cos\gamma_4)^{-\Delta}\,, \tag{3.27}$$

where C is the Gegenbauer polynomials and γ_4 is the angle between x and x'. Then the correlation function of the CFT in the real space can be written as

$$\begin{aligned}\langle \mathcal{O}(x)\mathcal{O}(x')\rangle &= \sum_p \langle \mathcal{O}(p)\mathcal{O}(-p)\rangle Y_p(x)Y_p^*(x') \\ &= \frac{2}{\pi^2}\frac{(\Delta-2)\Gamma(\Delta)}{\Gamma(\Delta-2)}\frac{1}{\sigma_E^{2\Delta}}\,,\end{aligned} \tag{3.28}$$

where $\sigma_E^2 = 2(1-\cos\gamma_4)$ is the invariant length of S^4. By making analytic continuation to lorentzian spacetime, we find the result agrees with (3.8).

Now we examine the nature of the CFT in de Sitter spacetime. Making analytic continuation to the lorentzian spacetime, the closed de Sitter spacetime

$$ds_4^2 = \frac{1}{H^2}(-dt^2 + \cosh^2 Ht\, d\Omega_3^2)\,, \tag{3.29}$$

is obtained and the invariant length becomes

$$\sigma_E^2 \to \frac{2}{H^2}\left(1+\sinh Ht \sinh Ht' - \cosh Ht \cosh Ht' \cos\gamma_3\right), \tag{3.30}$$

where γ_3 is the angle between (Ω_3, Ω_3'). For equal spatial point at which $\cos\gamma_3 = 1$, the two-point function is given by

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle \propto \left(\frac{H}{\sinh\left(\frac{Ht}{2}\right)}\right)^{2\Delta}. \tag{3.31}$$

We find this two-point function has periodicity in imaginary time $it \to it + 2\pi/H$. Thus we can interpret it as the thermal Green function with temperature $T = H/2\pi$.

4. CFT in FRW universe

We extend the calculation to FRW universe. Let us consider a coordinate transformation

$$z = f(u) - g(v), \qquad \tau = f(u) + g(v), \tag{4.1}$$

with arbitrary functions $f(u)$ and $g(v)$ where $u = t - y$ and $v = t + y$ [16]. Then the metric (2.14) becomes

$$l^{-2}ds^2 = e^{2\beta(y,t)}(dy^2 - dt^2) + e^{2\alpha(y,t)}(\delta_{ij}dx^i dx^j). \tag{4.2}$$

Here $\alpha(y,t)$ and $\beta(y,t)$ are given by

$$e^{2\beta(y,t)} = 4\frac{f'(u)g'(v)}{(f(u)-g(v))^2}, \qquad e^{2\alpha(y,t)} = \frac{1}{(f(u)-g(v))^2}. \tag{4.3}$$

The $y = 0$ slicing can be the FRW spacetime

$$l^{-2}ds_d^2 = -e^{2\beta_0(t)}dt^2 + e^{2\alpha_0(t)}\delta_{ij}dx^i dx^j, \tag{4.4}$$

with scale factor e^{α_0} where $\alpha_0(t) = \alpha(0,t)$ and $\beta_0(t) = \beta(0,t)$.

As in the de Sitter spacetime, some efforts to take the limit to the boundary are needed, because the FRW spacetime can be realized on only one slicing of the AdS spacetime, that is $y = 0$. We take the following procedure. First we take the limit $y \to 0$. Since $z(0,t) = e^{-\alpha_0(t)}$, further liming procedure is needed to go to the boundary $z \to 0$. The scale factor e^{α_0} has one integration of the constant. Then in general, the scale factor can be written as $e^{\alpha_0} = a(t)/a_\star$. If we take $a_\star \to 0$, $y = 0$ slicing effectively goes to the boundary $z(0,t) = a_\star/a(t) \to 0$. Thus we can take $\epsilon = a_\star$. Introducing the conformal time η, the d-dimensional line element is given by

$$ds_4^2 = (l/a_\star)^2 a(\eta)^2(-d\eta^2 + \delta_{ij}dx^i dx^j). \tag{4.5}$$

Thus in order to fix the scale factor of the d-dimensional spacetime, we should take $l \to 0$ as $a_\star \to 0$.

We use the invariant Green function in constructing the bulk-boundary propagator. The explicit form of the coordinate transformation is needed. The coordinate transformation $f(u)$ and $g(v)$ can be determined by specifying the scale factor $e^{\alpha_0(t)}$ and fixing the gauge $e^{\beta_0(t)}$. For example, we take $d = 4$ and

$$e^{-\alpha_0(t)} = f(t) - g(t) = a_\star t^{-\frac{2}{3(1+w)}}, \qquad e^{2\beta_0(t)} = 4\frac{f'(t)g'(t)}{(f(t)-g(t))^2} = 1, \tag{4.6}$$

where w is a constant parameter and can be regarded as the barotropic parameter of the matter which dominates the universe. Note that the conformal time is defined by

$$\eta = a_\star t^{-\frac{2}{3(1+w)}+1}\frac{3(1+w)}{1+3w}. \tag{4.7}$$

Thus, in order to fix the conformal time in the limiting procedure $a_\star \to 0$, we should take $t \gg 1$. The equations (4.6) give the simultaneous first order differential equations for $f(t)$ and $g(t)$;

$$\begin{aligned} f'(t) &= \frac{a_\star}{3(1+w)} t^{-\frac{5+3w}{3(1+w)}} \left(-1 + \frac{3}{2}(1+w)t\right), \\ f(t) &= g(t) + a_\star t^{-\frac{2}{3(1+w)}}, \end{aligned} \tag{4.8}$$

where we have used $t \gg 1$. Once the functions $f(t)$ and $g(t)$ are obtained, the coordinate transformation is determined by replacing $f(t) \to f(u)$ and $g(t) \to g(v)$ as

$$\begin{aligned} f(u) &= \frac{a_\star}{2} u^{-\frac{2}{3(1+w)}} \left(1 + \frac{3(1+w)}{1+3w} u\right), \\ g(v) &= \frac{a_\star}{2} v^{-\frac{2}{3(1+w)}} \left(-1 + \frac{3(1+w)}{1+3w} v\right). \end{aligned} \tag{4.9}$$

Then z and τ can be written in terms of y and t and we can show the following relations;

$$\begin{aligned} \lim_{a_\star \to 0, y \to 0} \tau &= \lim_{a_\star \to 0, y \to 0} (f(u) + g(v)) = a_\star t^{-2/3(1+w)+1} \frac{3(1+w)}{1+3w} = \eta, \\ \lim_{a_\star \to 0, y \to 0} \frac{\partial z(y,t)}{\partial y} &= - \lim_{a_\star \to 0, y \to 0} (f'(u) + g'(v)) = -a_\star t^{-2/3(1+w)} = -\frac{a_\star}{a(\eta)}. \end{aligned} \tag{4.10}$$

Using these relations, we obtain

$$\begin{aligned} \lim_{a_\star \to 0, y \to 0} \frac{\partial^2}{\partial y \partial y'} G(y, x; y' x') &= \lim_{a_\star \to 0, y \to 0} \left(\frac{\partial z}{\partial y}\frac{\partial z'}{\partial y'}\right) \frac{\partial^2}{\partial z \partial z'} G(y, x, ; y', x') \\ &= G_0 2^{2\Delta} \Delta^2 a_\star^{2\Delta} \left(\frac{a(\eta)^{-1} a(\eta')^{-1}}{-(\eta - \eta')^2 + |x^i - x'^i|^2}\right)^\Delta, \end{aligned} \tag{4.11}$$

where we have used the fact $\partial^2 G / \partial\tau\partial\tau'$ is the higher order of $a_\star$. Then using $\sqrt{-\gamma} = (a_\star/l)^{-d}\sqrt{-h}$ and $n^A(\partial/\partial X^A) = -\partial/\partial y$, (2.13) becomes

$$\begin{aligned} I[\phi_B] = \frac{1}{2} \int_{\mathcal{B}} d^d x d^d x' \sqrt{-h(x)} \sqrt{-h(x')} \phi_B(x) \phi_B(x') a_\star^{2(\Delta - d)} l^{d-1} \times \\ \times \left(G_0 2^{2\Delta} \Delta^2 \frac{a(\eta)^{-\Delta} a(\eta')^{-\Delta}}{|x - x'|^{2\Delta}}\right), \end{aligned} \tag{4.12}$$

where we denote $(a_\star/l)^2 ds_4^2 = h_{\mu\nu} dx^\mu dx^\nu$ and $|x - x'|^2 = -(\eta - \eta')^2 + |x^i - x'^i|^2$. Putting $a_\star = \epsilon$ and renormalizing ϕ_B as in the Minkowski case (2.18), we get

$$\langle \mathcal{O}(x) \mathcal{O}(x') \rangle_{\mathrm{FRW}} = C_2 \frac{a(\eta)^{-\Delta} a(\eta')^{-\Delta}}{|x - x'|^{2\Delta}}. \tag{4.13}$$

This two-point function is similar to the two-point function of quantum field in conformal vacuum. The two-point function $D(x,x)$ of conformally invariant field with conformal dimension Δ in conformally flat spacetime can be written as

$$D(x,x') = \Omega^{\Delta}(x) D_F(x,x') \Omega^{\Delta}(x') \,, \tag{4.14}$$

where $D_F(x,x')$ is the two-point function in flat spacetime and we denote the conformally flat spacetime as $h_{\mu\nu} = \Omega^{-2}\eta_{\mu\nu}$. The FRW universe is given by $\Omega(x) = a(\eta)^{-1}$. Comparing (2.19) and (4.13), we see the relation (4.14) actually holds with Δ as the conformal dimension.

As in the de Sitter spacetime, the CFT in various states can be obtained from bulk-boundary propagators with appropriate boundary conditions. The solution of the field equation in the bulk is written as

$$\phi(y,t,x^i) = \int dp\, G_{\mathcal{B}}(y,t,x^i;p)\phi_B(p) \,. \tag{4.15}$$

The bulk-boundary propagator $G_{\mathcal{B}}$ satisfies the field equation and

$$\lim_{y\to 0, a_\star=\epsilon} G_{\mathcal{B}}(y,t,x^i;p) = Y_p(t,x^i) \,, \tag{4.16}$$

where Y_p is the harmonics in FRW universe. Appropriate choice of the harmonics will lead to the CFT in desired vacuum state.

5. Discussions

We have developed a formalism to calculate the effective action of the strongly coupled CFT in curved spacetime from AdS/CFT correspondence. Using the fact that the de Sitter spacetime and FRW universe can be embedded into the AdS spacetime, the effective action of scalar fields was derived in those spacetime. Recently Hawking et. al. used the effective action derived via AdS/CFT correspondence and calculated the correction of tensor propagators by conformal matter during trace anomaly driven inflation [2]. Their calculations are limited to exactly de Sitter spacetime. Our formalism may be useful to extend their analysis to more realistic situations.

Further extensions of the formalism can be considered. First, AdS spacetime itself can be embedded in the higher dimensional AdS spacetime [17]. The coordinate transformation from Poincare coordinate

$$z = \eta\cos y, \quad x_1 = \eta \sin y \,, \tag{5.1}$$

gives

$$ds^2 = \left(\frac{l^2}{\cos^2 y}\right)\left(dy^2 + \frac{1}{\eta^2}(d\eta^2 - d\tau^2 + \delta_{ij}dx^i dx^j)\right) . \tag{5.2}$$

At $\cos y_0 = L^{-1}l$, the AdS_d with the curvature radius L is embedded. Thus, the CFT in AdS spacetime can be learned from the classical theory in the AdS bulk. So far, we investigate the pure AdS spacetime as the geometry of the bulk. However, more general spacetime with negative cosmological constant can be considered. It is known that

$$ds^2 = \left(\frac{l^2}{y^2}\right)\left(dy^2 + g_{\mu\nu}dx^\mu dx^\nu\right), \tag{5.3}$$

with any Ricci flat metric $g_{\mu\nu}$ and

$$ds^2 = \left(\frac{l^2}{\sinh^2 y}\right)\left(dy^2 + \sigma_{\mu\nu}dx^\mu dx^\nu\right), \tag{5.4}$$

with any metric $\sigma_{\mu\nu}$ with positive cosmological constant are solutions of $d+1$ dimensional spacetime with negative cosmological constant. Thus, it seems to be possible to extend the conjecture that classical theory in these bulk spacetime is dual to quantum field theory on its boundary. The interesting point is that the calculation done in section 3.2 can be straightforwardly extended because y dependence of the mode function is the same with the one in pure AdS spacetime. Because these spacetime are in general asymptotically non AdS spacetime, it would be challenging to identify the dual field theory.

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A. Derivation of the counter term

In this appendix, we present the derivation of the covariant form of the counter terms via Hamilton-Jacobi equation. Other derivations of the counter terms can be seen in [18] for pure gravity and in [19] for scalar field theory.

A.1 Hamilton-Jacobi equation

To derive the counter terms, we should know the behaviour of the classical action I at the boundary. One of the easiest way to obtain I is using the Hamilton-Jacobi (H-J) equation [20]. Let us consider the $(d+1)$-dimensional action

$$S = \int d^{d+1}x\sqrt{-g}\left(\frac{1}{2\kappa}\left({}^{(d+1)}R - 2\Lambda\right) - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right), \tag{A.1}$$

where ${}^{(d+1)}R$ is the Ricci tensor in $d+1$-dimensional spacetime. The line element is denoted as

$$ds^2 = (N^2 + \gamma_{\mu\nu} N^\mu N^\nu) du^2 + 2N_\mu du dx^\mu + \gamma_{\mu\nu} dx^\mu dx^\nu \,, \tag{A.2}$$

where N is the lapse function, N^μ is the shift function and $\gamma_{\mu\nu}$ is the d-dimensional metric. By defining the conjugate momentum

$$\pi^{\mu\nu} = \frac{\delta S}{\delta \dot{\gamma}_{\mu\nu}}, \quad \pi_\phi = \frac{\delta S}{\delta \dot{\phi}} \,, \tag{A.3}$$

the hamiltonian form of the action

$$S = \int d^{d+1}x \left(\pi^{\mu\nu} \dot{\gamma}_{\mu\nu} + \pi_\phi \dot{\phi} - N\mathcal{H} - N^\mu \mathcal{H}_\mu \right), \tag{A.4}$$

is obtained where

$$\begin{aligned}
\mathcal{H} &= -\frac{2\kappa}{\sqrt{-\gamma}} \pi^{\mu\nu} \pi^{\lambda\rho} \left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1} \gamma_{\mu\nu}\gamma_{\lambda\rho} \right) - \frac{\sqrt{-\gamma}}{2\kappa} (R - 2\Lambda) - \\
&\quad - \frac{1}{2\sqrt{-\gamma}} \pi_\phi^2 + \frac{1}{2} \sqrt{-\gamma} \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2\sqrt{-\gamma}} m^2 \phi^2 \,, \\
\mathcal{H}_\mu &= -2\pi_{\mu\,;\nu}^{\nu} + \pi_\phi \phi_{,\mu},
\end{aligned} \tag{A.5}$$

R is the Ricci tensor of $\gamma_{\mu\nu}$ and ; denotes the covariant derivative with respect to $\gamma_{\mu\nu}$. Variation of the action with respect to the momentum gives the equation of the motion for $\gamma_{\mu\nu}$ and ϕ;

$$\begin{aligned}
\dot{\gamma}_{\mu\nu} - N_{\mu;\nu} - N_{\nu;\mu} &= -N \frac{4\kappa}{\sqrt{-\gamma}} \pi^{\lambda\rho} \left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1} \gamma_{\mu\nu}\gamma_{\lambda\rho} \right), \\
\dot{\phi} - N^\mu \phi_{,\mu} &= -\frac{N}{\sqrt{-\gamma}} \pi_\phi \,,
\end{aligned} \tag{A.6}$$

where $\cdot$ denotes the derivative with respect to u. The H-J equation is obtained by putting

$$\pi^{\mu\nu} = \frac{\delta I}{\delta \gamma_{\mu\nu}} \,, \qquad \pi_\phi = \frac{\delta I}{\delta \phi} \,, \tag{A.7}$$

where I is the classical action of the system which is obtained by inserting the solutions of field equations into S. The H-J equation is given by

$$\begin{aligned}
\mathcal{H}\left(\frac{\delta I}{\delta \gamma_{\mu\nu}}, \frac{\delta I}{\delta \phi}, \gamma_{\mu\nu}, \phi \right) &= -\frac{2\kappa}{\sqrt{-\gamma}} \frac{\delta I}{\delta \gamma_{\mu\nu}} \frac{\delta I}{\delta \gamma_{\lambda\rho}} \left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1} \gamma_{\mu\nu}\gamma_{\lambda\rho} \right) - \frac{\sqrt{-\gamma}}{2\kappa} (R - 2\Lambda) - \\
&\quad - \frac{1}{2\sqrt{-\gamma}} \left(\frac{\delta I}{\delta \phi} \right)^2 + \frac{1}{2} \sqrt{-\gamma} \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \sqrt{-\gamma} \frac{1}{2} m^2 \phi^2 = 0 \,.
\end{aligned}$$

In the following calculations, we use the gauge $N = 1$ and $N^\mu = 0$. However, because the H-J equation does not contain neither Lapse function N nor shift function N_μ, the solution of the H-J equation does not depend on the choice of the slicing in $(d+1)$ spacetime.

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From (3.20) and (3.24), we notice that the expansion with respect to ϵ is also expansion with respect to the number of the derivatives of d-dimensional spacetime. Thus we will obtain I order by order with respect to the number of the derivatives of d-dimensional spacetime,

$$I = I^{(0)} + I^{(2)} + \cdots . \tag{A.8}$$

We will use fixed background approximation, that is, we assume the motion of the scalar field does not affects the geometry of the spacetime. We will treat separately a massless ($m = 0$) field and a massive $m > 0$ field.

A.2 Solution of H-J equation-massless case

The hamiltonian constraint can be also expanded in terms of the number of the derivatives as

$$\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(2)} + \cdots . \tag{A.9}$$

(1) 0-th order. At the 0-th order, the H-J equation is given by

$$\mathcal{H}^{(0)} = -\frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta I^{(0)}}{\delta\gamma_{\mu\nu}}\frac{\delta I^{(0)}}{\delta\gamma_{\lambda\rho}}\left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1}\gamma_{\mu\nu}\gamma_{\lambda\rho}\right) + \frac{\sqrt{-\gamma}}{\kappa}\Lambda = 0\,, \tag{A.10}$$

where we take $\delta I^{(0)}/\delta\phi = 0$ from fixed background approximation. Then putting the solution of the form

$$I^{(0)} = 2A\int d^dx\sqrt{-\gamma}\,, \tag{A.11}$$

we have

$$A = A_{\pm} = \pm\frac{d-1}{2\kappa l}\,, \tag{A.12}$$

where we have used $\Lambda = -d(d-1)/2l^2$. We will take the solution $A = A_+$. Inserting the solution into (A.7), we can obtain the solutions of the momentum. Then the equation of the motion (A.6) becomes

$$\dot{\gamma}_{\mu\nu} = \frac{2}{l}\gamma_{\mu\nu}\,. \tag{A.13}$$

Thus the solution is obtained as

$$\gamma_{\mu\nu} = \Omega^2(u)f_{\mu\nu}\,, \qquad \Omega(u) = e^{u/l}\,. \tag{A.14}$$

where $f_{\mu\nu}$ is independent of u.

(2) 2-nd order. The H-J equation is given by

$$\mathcal{H}^{(2)} = \frac{2}{l}\gamma_{\mu\nu}\frac{\delta I^{(2)}}{\delta\gamma_{\mu\nu}} - \frac{\sqrt{-\gamma}}{2\kappa}R + \frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\nu\phi\partial_\mu\phi = 0\,. \tag{A.15}$$

Using the solution $(A.14)$, we can write

$$\frac{\delta I^{(2)}}{\delta u} = \frac{2}{l}\gamma_{\mu\nu}\frac{\delta I^{(2)}}{\delta\gamma_{\mu\nu}}\,. \tag{A.16}$$

By the standard result of the conformal transformation

$$R(\gamma_{\mu\nu}) = \Omega^{-2}R(f_{\mu\nu})\,, \qquad \sqrt{-\gamma} = \Omega^d\sqrt{-f}\,, \tag{A.17}$$

the H-J equation becomes

$$\begin{aligned}\frac{\delta I^{(2)}}{\delta u} &= \frac{1}{2\kappa}\sqrt{-\gamma}\left(R(\gamma_{\mu\nu}) - \kappa\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\\ &= \frac{1}{2\kappa}\sqrt{-f}\left(R(f_{\mu\nu}) - \kappa f^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)\Omega(u)^{d-2}\,.\end{aligned} \tag{A.18}$$

Apart from $\Omega(u)^{d-2}$, the right hand side of the equation is independent of u. Thus we can immediately integrate the equation. The solution is given by

$$I^{(2)} = \frac{l}{d-2}\int d^dx\sqrt{-\gamma}\left(\frac{1}{2\kappa}R - \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right). \tag{A.19}$$

(2) 4-th order. The H-J equation is given by

$$\begin{aligned}\mathcal{H}^{(4)} &= \frac{2}{l}\gamma_{\mu\nu}\frac{\delta I^{(4)}}{\delta\gamma_{\mu\nu}} - \frac{2\kappa}{\sqrt{-\gamma}}\frac{\delta I^{(2)}}{\delta\gamma_{\mu\nu}}\frac{\delta I^{(2)}}{\delta\gamma_{\lambda\rho}}\left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1}\gamma_{\mu\nu}\gamma_{\rho\lambda}\right) - \frac{1}{2\sqrt{-\gamma}}\left(\frac{\delta I^{(2)}}{\delta\phi}\right)^2\\ &= 0\,.\end{aligned} \tag{A.20}$$

Inserting the solution (A.14) and $I^{(2)}$ (A.19), the H-J equation becomes

$$\begin{aligned}\frac{\delta I^{(4)}}{\delta u} &= \Omega(u)^{d-4}F_4[f,\phi]\,,\\ F_4[f,\phi] &= \frac{l^2}{2\kappa(d-2)^2}\sqrt{-f}\Bigg(R_{\mu\nu}R^{\mu\nu} - \frac{d}{4(d-1)}R^2 + \frac{d}{2(d-1)}\kappa R\partial_\mu\phi\partial^\mu\phi -\\ &\qquad - 2\kappa R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \kappa(\Box\phi)^2 + \frac{3d-4}{4(d-1)}\kappa^2(\partial_\mu\phi\partial^\mu\phi)^2\Bigg)\,.\end{aligned}$$

Again, we observe that right-hand side of the equation is independent of u apart from $\Omega(u)^{d-4}$. Then we can easily integrate the equation to obtain

$$I^{(4)} = \frac{l^3}{2\kappa(d-2)^2(d-4)}\int d^dx\sqrt{-\gamma}F_4[\gamma,\phi]\,. \tag{A.21}$$

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(3)higher order. The higher order solution $\mathcal{H}^{(2n)}$ $(n \geq 2)$ can be obtained recursively. The H-J equation can be written as

$$
\begin{aligned}
\mathcal{H}^{(2n)} &= \frac{\delta I^{(2n)}}{\delta u} - F_{2n}[\gamma, \phi] = 0\,, \\
F_{2n}[\gamma, \phi] &= +\frac{2\kappa}{\sqrt{-\gamma}} \sum_{p=1}^{n-1} \frac{\delta I^{(2p)}}{\delta \gamma_{\mu\nu}} \frac{\delta I^{(2n-2p)}}{\delta \gamma_{\lambda\rho}} \left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1}\gamma_{\mu\nu}\gamma_{\rho\lambda} \right) + \\
&\quad + \frac{1}{2\sqrt{-\gamma}} \sum_{p=1}^{n-1} \frac{\delta I^{(2p)}}{\delta \phi} \frac{\delta I^{(2n-2p)}}{\delta \phi}\,.
\end{aligned} \tag{A.22}
$$

Using the solution $(A.14)$, we find $F_{2n}[\gamma, \phi] = \Omega^{d-2n}(u) F_{2n}[f, \phi]$. Then integration of the equation yields

$$
I^{(2n)} = \frac{l}{d-2n} F_{2n}[\gamma, \phi]\,. \tag{A.23}
$$

A.3 Solution of H-J equation -massive case

(1) 0-th order. We take the following ansatz for 0-th order solution

$$
I^{(0)} = 2 \int d^d x \sqrt{-\gamma} H(\phi)\,. \tag{A.24}
$$

The H-J equation is gives by

$$
2\kappa \frac{d}{d-1} H^2(\phi) - 2 \left(\frac{\partial H(\phi)}{\partial \phi} \right)^2 + \frac{\Lambda}{\kappa} + \frac{1}{2} m^2 \phi^2 = 0\,. \tag{A.25}
$$

From the background approximation, we assume

$$
H(\phi) = A + \frac{1}{2} B \phi^2\,, \tag{A.26}
$$

where $A \gg B\phi^2$ and A is given by (A.12). Then we obtain

$$
B = \frac{d \pm \sqrt{d^2 + 4m^2 l^2}}{4l} \equiv \frac{d - \Delta_{\mp}}{2l}\,, \tag{A.27}
$$

We will pick the solution $\Delta_+ = \Delta$. The equation of motion $(A.6)$ becomes

$$
\begin{aligned}
\dot{\gamma} &= \frac{2}{l} \gamma_{\mu\nu}\,, \\
\dot{\phi} &= -\frac{d-\Delta}{l} \phi\,.
\end{aligned} \tag{A.28}
$$

The solution can be obtained as

$$
\begin{aligned}
\gamma_{\mu\nu} &= \Omega^2(u) f_{\mu\nu}, \qquad & \Omega(u) &= e^{u/l}\,, \\
\phi &= \Pi(u) C\,, \qquad & \Pi(u) &= e^{-(d-\Delta)u/l}\,.
\end{aligned} \tag{A.29}
$$

where C is independent of u. In the following, we divide the classical action as

$$
I = I_0(\gamma_{\mu\nu}) + I_1(\gamma_{\mu\nu}, \phi)\,, \tag{A.30}
$$

and assume $I_0 \gg I_1$.

(2)2-nd order. Inserting the solution of the first order, the H-J equation becomes

$$\frac{2}{l}\gamma_{\mu\nu}\frac{\delta I_0^{(2)}}{\delta\gamma_{\mu\nu}} - \sqrt{-\gamma}\frac{R}{2\kappa} = 0\,,$$
$$\frac{2}{l}\gamma_{\mu\nu}\frac{\delta I_1^{(2)}}{\delta\gamma_{\mu\nu}} - \frac{d-\Delta}{l}\phi\frac{\delta I_1^{(2)}}{\delta\phi} = -\frac{1}{2}\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{d-\Delta}{4(d-1)}\sqrt{-\gamma}R\phi^2\,. \quad (A.31)$$

The equation for $I_0^{(2)}$ is the same with (A.15) without ϕ dependent term. Then $I_0^{(2)}$ is given by

$$I_0^{(2)} = \frac{l}{2\kappa(d-2)}\int d^dx\sqrt{-\gamma}R\,. \quad (A.32)$$

We can use the solution $(A.29)$ to write

$$\frac{\delta I_1^{(2)}}{\delta u} = \frac{2}{l}\gamma_{\mu\nu}\frac{\delta I_1^{(2)}}{\delta\gamma_{\mu\nu}} - \frac{d-\Delta}{l}\phi\frac{\delta I_1^{(2)}}{\delta\phi}\,. \quad (A.33)$$

Then the H-J equation gives

$$\frac{\delta I_1^{(2)}}{\delta u} = -\left[\frac{d-\Delta}{4(d-1)}\sqrt{-f}R(f)C^2 + \frac{1}{2}\sqrt{-f}f^{\mu\nu}\partial_\mu C\partial_\nu C\right]\Omega(u)^{d-2}\Pi(u)^2\,. \quad (A.34)$$

Again apart from the term $\Omega(u)^{d-2}\Pi(u)^2$, the right-hand side is independent of u. Then integration can be done to give

$$I_1^{(2)} = -\frac{l}{2\Delta-d-2}\int d^dx\sqrt{-\gamma}\left(\frac{d-\Delta}{4(d-1)}R\phi^2 + \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right). \quad (A.35)$$

(3) higher order. The H-J equation for $I_0^{(2n)}$ is the same with (A.23) without ϕ dependent terms

$$I_0^{(2n)} = \frac{l}{d-2n}F_{2n}[\gamma]\,. \quad (A.36)$$

Then the solution can be found easily. By using the solution (A.29), the H-J equation for $I_1^{(2n)}$ is written as

$$\mathcal{H}_1^{(2n)} = \frac{\delta I_1^{(2n)}}{\delta u} - G_{2n}[\gamma,\phi] = 0\,,$$
$$G_{2n}[\gamma,\phi] = \frac{4\kappa}{\sqrt{-\gamma}}\sum_{p=1}^{n}\frac{\delta I_0^{(2p)}}{\delta\gamma_{\mu\nu}}\frac{\delta I_1^{(2n-2p)}}{\delta\gamma_{\lambda\rho}}\left(\gamma_{\mu\rho}\gamma_{\nu\lambda} - \frac{1}{d-1}\gamma_{\mu\nu}\gamma_{\rho\lambda}\right) +$$
$$+\frac{1}{2\sqrt{-\gamma}}\sum_{p=1}^{n-1}\frac{\delta I_1^{(2p)}}{\delta\phi}\frac{\delta I_1^{(2n-2p)}}{\delta\phi}\,. \quad (A.37)$$

Using the solution (A.29), we find $G_{2n}[\gamma,\phi] = \Omega(u)^{d-2n}\Pi(u)^2G_{2n}[f,C]$. Then integration of the equation gives

$$I^{(2n)} = \frac{l}{2\Delta-d-2n}G_{2n}[\gamma,\phi]\,. \quad (A.38)$$

A.4 counter term

The counter terms are defined by

$$S^{(0)} + S^{(2)} + \cdots = -(I^{(0)} + I^{(2)} + \cdots)\,. \tag{A.39}$$

We summarise the result up to terms which are necessary for the calculation with $d = 4$;

1. massless case

$$\begin{aligned}
S^{(0)} &= -\frac{d-1}{\kappa l}\int d^d x\sqrt{-\gamma}\,,\\
S^{(2)} &= -\frac{l}{d-2}\int d^d x\sqrt{-\gamma}\left(\frac{1}{2\kappa}R - \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right),\\
S^{(4)} &= -\frac{l^3}{2(d-2)^2(d-4)}\int d^d x\sqrt{-\gamma}\left(\frac{1}{\kappa}\left(R_{\mu\nu}R^{\mu\nu} - \frac{d}{4(d-1)}R^2\right) + \right.\\
&\quad \left. + \frac{d}{2(d-1)}R\partial_\mu\phi\partial^\mu\phi - 2R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (\Box\phi)^2 + \frac{3d-4}{4(d-1)}\kappa(\partial_\mu\phi\partial^\mu\phi)^2\right).
\end{aligned} \tag{A.40}$$

2. massive case

$$\begin{aligned}
S^{(0)} &= -\frac{d-1}{\kappa l}\int d^d x\sqrt{-\gamma} - \frac{d-\Delta}{2l}\int d^d x\sqrt{-\gamma}\phi^2\,,\\
S^{(2)} &= -\frac{l}{d-2}\int d^d x\sqrt{-\gamma}\frac{1}{2\kappa}R + \frac{l}{2\Delta-d-2}\int d^d x\sqrt{-\gamma}\,\times\\
&\quad\times\left(\frac{d-\Delta}{4(d-1)}R\phi^2 + \frac{1}{2}\gamma^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right).
\end{aligned} \tag{A.41}$$

Let us take $d = 4$. For massless case, $S^{(4)}$ diverges at $d = 4$. We replace $1/(d-4)$ by $-\log\epsilon$. The counter terms can be rewritten using the metric of d-dimensional spacetime $h_{\mu\nu}$ which is defined as (see (3.6)),

$$\gamma_{\mu\nu} = \left(\frac{l}{\epsilon}\right)^2 h_{\mu\nu}\,. \tag{A.42}$$

Rescaling the field as $\phi \to l^{-3/2}\phi$, we obtain

1. massless case

$$\begin{aligned}
S^{(2)} &= \frac{1}{4}\epsilon^{-2}\int d^4 x\sqrt{-h}h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\,,\\
S^{(4)} &= \frac{1}{8}\log\epsilon\int d^4 x\sqrt{-h}\left(\frac{2}{3}R\partial_\mu\phi\partial^\mu\phi - 2R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + (\Box\phi)^2\right),
\end{aligned} \tag{A.43}$$

2. massive case

$$S^{(0)} = -\frac{1}{2}\epsilon^{-4}\int d^4x\sqrt{-h}(4-\Delta)\phi^2\,,$$
$$S^{(2)} = \frac{1}{4(\Delta-3)}\epsilon^{-2}\int d^4x\sqrt{-h}\left(\frac{4-\Delta}{6}R\phi^2 + h^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right), \qquad \text{(A.44)}$$

where R, $R_{\mu\nu}$ and $\Box$ are defined in terms of $h_{\mu\nu}$.

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